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On Divided Rings and ϕ-Pseudo-Valuation Rings

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Abstract. Let R be a commutative ring with 1 and $T(R)$ be its total quotient ring such that $Nil(R)$ is a divided prime ideal of R. Then R is called a *ϕ-chained ring (ϕ-CR)* if for every $x, y \in R \setminus Nil(R)$ either $x \mid y$ or $y \mid x$. Also, R is called a *ϕ-pseudo-valuation ring (ϕ-PVR)* if for every $x, y \in R \setminus Nil(R)$ either $x \mid y$ or $y \mid xm$ for each nonunit $m \in R$. We show that a ring R is a ϕ-PVR iff $Nil(R)$ is a divided prime ideal and $R/Nil(R)$ is a pseudo-valuation domain. Also, we show that every overring of a quasi-local ring R with maximal ideal M is a ϕ-PVR iff $R[u]$ is quasilocal for each $u \in (M : M) \setminus R$ iff every overring of R is quasilocal iff every ϕ-CR between R and $T(R)$ other than $(M : M)$ is of the form R_P for some nonmaximal prime ideal P of R. Among other results, we show that if B is an overring of a ϕ-PVR and I is a proper ideal of B, then there is a ϕ-CR C between B and $T(R)$ such that $IC \neq C$. Also, we show that the integral closure R' of R in $T(R)$ is the intersection of all the ϕ-CRs between R and $T(R)$.

1. Introduction

We assume throughout that all rings are commutative with $1 \neq 0$. We begin by recalling some background material. As in [15], an integral domain R, with quotient field K, is called a *pseudo-valuation domain (PVD)* in case each prime ideal P of R is *strongly prime*, in the sense that $xy \in P, x \in K, y \in K$ implies that either $x \in P$ or $y \in P$. In [5], Anderson, Dobbs and the author generalized the study of pseudo-valuation domains to the context of arbitrary rings (possibly with nonzero zerodivisors). Recall from [5] that a prime ideal P of R is said to be *strongly prime (in R)* if aP and bR are comparable (under inclusion) for all $a, b \in R$. A ring R is called a *pseudo-valuation ring (PVR)* if each prime ideal of R is strongly prime. A PVR is necessarily quasilocal [5, Lemma 1(b)]; a chained ring is a PVR [5, Corollary 4]; and an integral domain is a PVR if and only if it is a PVD (cf. [1, Proposition 3.1], [2, Proposition 4.2], and [6, Proposition 3]). Recall from [7] and [13] that a prime ideal P of R is called *divided* if it is comparable (under inclusion) to every ideal of R. A ring R is called a *divided ring* if every prime ideal of R is divided.

In [8], the author gave another generalization of PVDs to the context of arbitrary rings (possibly with nonzero zerodivisors). As in [8], for a ring R with total quotient ring $T(R)$ such that $Nil(R)$ is a divided prime ideal of R, let $\phi : T(R) \longrightarrow K := R_{Nil(R)}$ such that $\phi(a/b) = a/b$ for every $a \in R$ and every $b \in R \setminus Z(R)$. Then ϕ is a ring homomorphism from $T(R)$ into K, and ϕ restricted to R is also a ring homomorphism from R into K given by $\phi(x) = x/1$ for every $x \in R$. A prime ideal Q of $\phi(R)$ is called a *K-strongly prime* ideal if $xy \in Q$, $x \in K, y \in K$ implies that either $x \in Q$ or $y \in Q$. If each prime ideal of $\phi(R)$ is K-strongly prime, then $\phi(R)$

is called a *K-pseudo-valuation ring (K-PVR)*. A prime ideal P of R is called a *ϕ-strongly prime* ideal if $\phi(P)$ is a K-strongly prime ideal of $\phi(R)$. If each prime ideal of R is ϕ-strongly prime, then R is called a *ϕ-pseudo-valuation ring ($\phi-PVR$)*. For an equivalent characterization of a ϕ-PVR, see Proposition 1.1(5). It was shown in [9, Theorem 2.6] that for each $n \geq 0$ there is a ϕ-PVR of Krull dimension n that is not a PVR. Also, recall from [10], that a ring R is called a *ϕ-chained ring (ϕ-CR)* if $Nil(R)$ is a divided prime ideal of R and for every $x \in R_{Nil(R)} \setminus \phi(R)$, we have $x^{-1} \in \phi(R)$. For an equivalent characterization of a ϕ-CR, see Lemma 3.9. A ϕ-CR is a divided ring [10, Corollary 3.3(2)], and hence is quasilocal. It was shown in [10, Theorem 2.7] that for each $n \geq 0$ there is a ϕ-CR of Krull dimension n that is not a chained ring.

In this paper, we show that a quasilocal ring R with maximal ideal M is a ϕ-PVR iff $R[u]$ is quasilocal for each $u \in (M : M) \setminus R$ iff every overring of R is quasilocal iff every overring contained in $(M : M)$ is quasilocal iff each ϕ-CR between R and $T(R)$ other than $(M : M)$ is of the form R_P for some nonmaximal prime ideal P of R. Among other results, we show that if B is an overring of a ϕ-PVR and I is a proper ideal of B, then there is a ϕ-CR C between B and $T(R)$ such that $IB \neq B$. Also, we show that the integral closure of R in $T(R)$ is the intersection of all the ϕ-CRs between R and $T(R)$.

The following notations will be used throughout. Let R be a ring. Then $T(R)$ denotes the total quotient ring of R, $Nil(R)$ denotes the set of nilpotent elements of R, and $Z(R)$ denotes the set of zerodivisors of R. If I is an ideal of R, then $Rad(I)$ denotes the radical ideal of $I(inR)$.

We summarize some basic properties of PVRs and ϕ-PVRs in the following proposition.

PROPOSITION 1.1.
1. *A PVR is a divided ring* [5, Lemma 1], *and hence is quasilocal.*
2. *A ϕ-PVR is a divided ring* [8, Proposition 4], *and hence is quasilocal.*
3. *An integral domain is a PVR iff it is a ϕ-PVR iff it is a PVD(* [1, Proposition 3.1], [2, Proposition 4.2], [6, Proposition 3], *and* [8]*).*
4. *A ring R is a PVR if and only if for every $a, b \in R$, either $a \mid b$ in R or $b \mid ac$ in R for each nonunit c in R* [5, Theorem 5].
5. *A ring R is a ϕ-PVR if and only if $Nil(R)$ is a divided prime ideal of R and for every $a, b \in R \setminus Nil(R)$, either $a \mid b$ in R or $b \mid ac$ in R for every nonunit $c \in R$* [8, Corollary 7(2)].
6. *If R is a PVR or a ϕ-PVR, then $Nil(R)$ and $Z(R)$ are divided prime ideals of R (*[5], [8]*).*

2. Divided Rings and ϕ-PVRs

Definition. *A proper ideal I of a ring R is called a divided ideal if I is comparable (under inclusion) to every principal ideal of R; equivalently, if I is comparable to every ideal of R. If every prime ideal of R is divided, then R is called a divided ring.*

In view of the proof of [9, Proposition 2.1], we see that the result in [9, Proposition 2.1] is still valid if we only assume that the ring D is a divided domain. Hence, we state the following result without proof.

PROPOSITION 2.1. [9, Proposition 2.1] *Let D be a divided domain with maximal ideal M and Krull dimension n, say $M = P_n \supset P_{n-1} \supset \ldots \supset P_1 \supset \{0\}$, where the P_i's are the distinct prime ideals of D. Let $i, m, d \geq 1$ such that $1 \leq i \leq m \leq n$. Choose $x \in D$ such that $Rad((x)) = P_i$. Let $Q := P_m$ and $J := x^{d+1}D_Q$. Then:*

1. *J is an ideal of D and $Rad(J) = P_i$.*
2. *$R := D/J$ is a divided ring with maximal ideal M/J, $Z(R) = P_m/J$, and $Nil(R) = P_i/J$. Furthermore, $w := x + J \in Nil(R)$ and $w^d \neq 0$ in R.*
3. *$dim(R) = n - i$.*
4. *If $i < m < n$, then $Nil(R)$ is properly contained between $Z(R)$ and M/J.*

Recall that a prime ideal P of a ring A is called branched if $Rad(I) = P$ for some primary ideal $I \neq P$ of A. It is well-known that a prime ideal P of a Prüfer domain D is branched iff $Rad(I) = P$ for some ideal $I \neq P$ of D. In the following result we will show that this result is still valid for divided rings.

PROPOSITION 2.2. *Let R be a divided ring, and let P be a prime ideal of R such that $P \neq Nil(R)$. Then P is branched if and only if $Rad(I) = P$ for some ideal $I \neq P$ of R.*

Proof. Suppose that $Rad(I) = P$ for some ideal $I \neq P$ of R. It is clear that $Rad(IP) \subset P$. Let $x \in P$. Since $Rad(I) = P$, $x^n \in I$ for some $n \geq 1$. Hence, $x^{n+1} \in IP$. Thus, $P \subset Rad(IP)$. Now, we show that IP is a primary ideal of R. Suppose that $xy \in IP$ for some $x, y \in R$ and $x \notin P$. Since $xy \in IP$, $xy = i_1p_1 + \ldots + i_np_n$, where each $i_k \in I$ and each $p_k \in P$, $1 \leq k \leq n$. Since P is a divided prime ideal and $x \notin P$, $p_k = q_kx$ for some $q_k \in P$ for each $p_k \in P$. Thus, $x[y - (i_1q_1 + \ldots + i_nq_n)] = 0$. Since $Nil(R)$ is a prime ideal of R and $x \notin Nil(R)$, $y - (i_1q_1 + \ldots + i_nq_n) = w \in Nil(R)$. Since $Rad(IP) = P \neq Nil(R)$, there is a $d \in IP \setminus Nil(R)$. Hence, $Nil(R) \subset (d) \subset IP$. Since $i_1q_1 + \ldots + i_nq_n \in IP$ and $w \in Nil(R) \subset IP$, $y \in IP$. Thus, IP is a primary ideal of R. □

In light of the proof of the above proposition, we have the following corollary.

COROLLARY 2.3. *Let R be a ring such that $Nil(R)$ is a divided prime ideal of R, and let P be a divided prime ideal of R such that $P \neq Nil(R)$. Then P is branched if and only if $Rad(I) = P$ for some ideal $I \neq P$ of R.*

PROPOSITION 2.4. *Let R be a ring such that $Nil(R)$ is a divided prime ideal of R. Suppose that I is a proper ideal of R such that I contains a nonnilpotent of R and for some $N \geq 1$, I^n is a divided ideal of R for each $n \geq N$. Then $P = \cap_{n\geq1} I^n$ is a divided prime ideal of R.*

Proof. Since $Nil(R)$ is a divided ideal and I contains a nonnilpotent of R, $Nil(R) \subset I^n$ for each $n \geq N$. Hence, $Nil(R) \subset P$. Now, suppose that $xy \in P$ for some $x, y \in R$ and suppose that $x \notin P$. Hence, $x \notin I^m$ for some $m \geq N$. Hence, $I^m \subset (x)$. Thus, for each $k \geq 1$ we have $xy \in I^{m+k} \subset xI^k$. Hence, for each $k \geq 1$, there is a $d_k \in I^k$ such that $xy = xd_k$. Thus, $x(y - d_k) = 0$ for each $k \geq 1$. Since $Nil(R) \subset P$ and $Nil(R)$ is a prime ideal of R and $x \notin P$, we have $y - d_k = w_k \in Nil(R)$. Hence, $y = d_k + w_k \in I^k$. Hence, $y \in P$. Thus, P is a prime ideal of R. Now, we show that P is divided. Let $x \notin P$. Then $x \notin I^m$ for some $m \geq N$. Hence, $P \subset I^m \subset (x)$. □

In view of the above proposition, we have the following corollary.

COROLLARY 2.5. *Let R be a ring such that $Nil(R)$ is a divided prime ideal of R, and let I be a proper ideal of R such that I contains a nonnilpotent of R. Then the following statements are equivalent:*

1. *$I^n = I^m$ for some positive integers $n \neq m$ and I^n is a divided ideal of R.*
2. *I is a divided prime ideal of R and $I = I^2$.*

In the following result, we give a characterization of ϕ-PVRs in terms of divided ideals.

PROPOSITION 2.6. *Let R be a quasilocal ring with maximal ideal M. Then the following statements are equivalent:*

1. *R is a ϕ-PVR.*
2. *aM is a divided ideal of R for each $a \in R \setminus Nil(R)$.*

Proof. **(1)⇒ (2).** Let $a \in R \setminus Nil(R)$ and $b \notin aM$. If b = ar for some unit r of R, then $b \mid am$ for every $m \in M$. Otherwise, $a \nmid b$ in R. Thus, $b \mid am$ for each $m \in M$ by Proposition 1.1(5). Hence, $aM \subset (b)$.
(2)⇒ (1). Let $w \in Nil(R)$ and $a \in R \setminus Nil(R)$. If a is a unit of R, then $a \mid w$ in R. Hence, assume that a is a nonunit of R. Since $w \nmid a^2$, $aM \not\subset (w)$. Hence, $w \in aM$. Thus, $a \mid w$. Thus, $Nil(R)$ is a divided ideal of R. Hence, $Nil(R)$ is a prime ideal of R by [3, Proposition 5.1]. Now, let $a, b \in R \setminus Nil(R)$. Then either $b \in aM$ or $aM \subset (b)$. Hence, either $a \mid b$ or $b \mid am$ for each $m \in M$. Thus, R is a ϕ-PVR by Proposition 1.1(5). □

The following result follows directly from the definition of strongly prime ideal as in [5] and the fact that a quasilocal ring with maximal ideal M is a PVR if and only if M is strongly prime [5, Theorem 2].

PROPOSITION 2.7. *For a quasilocal ring R with maximal ideal M, the following statements are equivalent:*

1. *R is a PVR.*
2. *aM is a divided ideal for each $a \in M$.*

An element d in a ring R is called a proper divisor of $s \in R$ if $s = dm$ for some nonunit $m \in R$. The proof of the following result is very similar to that in [11, Proposition 4], but here we make use of the above proposition.

PROPOSITION 2.8. *A ring R is a ϕ-PVR if and only if $Nil(R)$ is a divided prime ideal of R and for every $a, b \in R \setminus Nil(R)$, either $b \mid a$ in R or $d \mid b$ in R for each proper divisor d of a.*

Proof. Suppose that R is a ϕ-PVR with maximal ideal M. Then $Nil(R)$ is a divided prime ideal of R by Proposition 1.1(6). Let $a, b \in R \setminus Nil(R)$, and suppose that $b \nmid a$ in R. Let d be a proper divisor of a. Since $Nil(R)$ is a divided ideal of R and $b \nmid a$, we conclude that $d \notin Nil(R)$. Thus, since dM is a divided ideal by Proposition 2.6 and $b \nmid a$ in R, $b \in dM$. Conversely, suppose that $b \nmid a$ in R for some $a, b \in R \setminus Nil(R)$. We need to show that $a \mid bm$ for each nonunit $m \in R$. Suppose that $a \nmid bm$ for some nonunit $m \in R$. Since b is a proper divisor of bm, $b \mid a$ which is a contradiction. Hence, $a \mid bm$ for each nonunit $m \in R$. Thus, R is a ϕ-PVR by Proposition 1.1(5). □

In the following result, we make a connection between ϕ-PVR's and PVR's.

PROPOSITION 2.9. *A ring R is a ϕ-PVR if and only if $Nil(R)$ is a divided prime ideal of R and $R/Nil(R)$ is a PVR.*

Proof. Suppose that R is a ϕ-PVR. Then $Nil(R)$ is a divided prime ideal of R by Proposition 1.1(6). By applying Proposition 1.1(4) to the ring $R/Nil(R)$, one can conclude that $R/Nil(R)$ is a PVR. Conversely, suppose that $Nil(R)$ is a divided prime ideal of R and $R/Nil(R)$ is a PVR. Let $a, b \in R \setminus Nil(R)$ and c be a nonunit of R. Then it is easy to see that $c + Nil(R)$ is a nonunit of $R/Nil(R)$. Hence, by Proposition 1.1(4) either $a + Nil(R) \mid b + Nil(R)$ in $R/Nil(R)$ or $b + Nil(R) \mid ac + Nil(R)$ in $R/Nil(R)$. Suppose that $a + Nil(R) \mid b + Nil(R)$ in $R/Nil(R)$. Then $b = ak + w$ in R for some $w \in Nil(R)$ and $k \in R$. Since $Nil(R)$ is a divided prime ideal of R and $a \notin Nil(R)$, $a \mid w$. Thus, $a \mid b$ in R. Now, assume that $b + Nil(R) \mid ac + Nil(R)$ in $R/Nil(R)$. Then by an argument similar to the one just given we conclude that $b \mid ac$ in R. Thus, R is a ϕ-PVR by Proposition 1.1(5). □

3. ϕ-PVRs and ϕ-CRs

Let VD denote a valuation domain and CR denote a chained ring. We then have the following implications, none of which are reversible.

$$VD \Rightarrow PVD \Rightarrow PVR \Rightarrow \phi - PVR$$
AND
$$VD \Rightarrow CR \Rightarrow \phi - CR \Rightarrow \phi - PVR.$$

We start with the following lemma.

LEMMA 3.1. *Let R be a ϕ-PVR, and let P be a prime ideal of R. Then $x^{-1}P \subset P$ for each $x \in T(R) \setminus R$.*

Proof. Let $x = a/b \in T(R) \setminus R$ for some $a \in R$ and for some $b \in R \setminus Z(R)$. Since $b \nmid a$ in R and $Z(R)$ is a divided prime ideal by Proposition 1.1(6), we conclude that $a \in R \setminus Z(R)$. Hence, $x^{-1} = b/a \in T(R)$. Now, let $p \in P$. Then $x(x^{-1}p) = p \in P$. Hence, $\phi(xx^{-1}p) = \phi(x)\phi(x^{-1}p) = \phi(p) \in \phi(P)$. Since $\phi(P)$ is a K-strongly prime ideal of $\phi(R)$ and by [8, Proposition 3(3)] $\phi(x) \notin \phi(P)$, we conclude that $\phi(x^{-1}p) \in \phi(P)$. Thus, $\phi(x^{-1}p) = \phi(q)$ for some $q \in P$. Hence, $x^{-1}p - q \in Ker(\phi)$. Since $q \in P$, $Ker(\phi) \subset Nil(R)$ by [8, Proposition 2(1)], and $Nil(R) \subset P$, we conclude that $x^{-1}p \in P$. □

PROPOSITION 3.2. *Let R be a ϕ-PVR and $z \in T(R) \setminus R$ be integral over R. Then there is a minimal monic polynomial $f(x) \in R[x]$ such that $f(z) = 0$ and all nonzero coefficients of $f(x)$ are units in R. Furthermore, if $g(x)$ is a minimal monic polynomial in $R[x]$ such that $g(z) = 0$, then $g(0)$ is a unit in R.*

Proof. Let $g(x)$ be a minimal monic polynomial in $R[x]$ such that $g(z) = 0$. Suppose that a_0, the constant term of $g(x)$, is a nonunit of R. Since $z \in T(R) \setminus R$ is integral over R, $z^{-1} \notin R$. Hence, by Lemma 3.1, $z^{-1}a_0 = m$ is a nonunit of R. Thus, $mz = a_0$. Hence, we can replace the constant term a_0 in $g(x)$ with mx. Thus, we may factor x from all terms of $g(x)$ and get a monic polynomial $H(x)$ of less degree than $g(x)$ such that $H(z) = 0$, a contradiction. Hence, a_0 is a unit in R. Now, assume that c_kx^k is a term in $g(x)$ such that c_k is a nonunit of R. Since z^k is integral over R, $z^{-k} \notin R$. Hence, by Lemma 3.1, $c_kz^k = s$ is a nonunit of R. Thus, we may replace the term c_kx^k in $g(x)$ with s. Since s is a nonunit of R and a_0 is a unit in R and R is quasilocal, $s + a_0$ is a unit in R. Continuing in this manner, we get a minimal monic polynomial $f(x)$ such that $f(z) = 0$ and all nonzero coefficients of $f(x)$ are units in R. The remaining part of the Proposition follows directly from the first part of our proof. □

It is well-known ([16],[5],[8], [11]) that the integral closure of a PVR is a PVR. In view of the above result, one can give an alternative proof of this fact. For a ring R, let R' denotes the integral closure of R in $T(R)$.

PROPOSITION 3.3. *Let R be a ϕ-PVR with maximal ideal M, and let B be an overring of R such that $B \subset R'$. Then B is a ϕ-PVR with maximal ideal M.*

Proof. Let $x \in B \setminus R$. Hence, $x^{-1} \in R'$ by Proposition 3.2. Thus, $x^{-1} \in R[x] \subset B$ by [18, Theorem 15]. Hence, x is a unit in B. Since $1/s$ is never integral over R for any $s \in M$ and any $x \in B \setminus R$ is a unit in B, M is the maximal ideal of B. Thus, by applying Proposition 1.1(5) to the ring B, we conclude that B is a ϕ-PVR with maximal ideal M. □

PROPOSITION 3.4. *Let R be a ϕ-PVR with maximal ideal M, and let B be an overring of R. Then the following statements are equivalent:*

1. *$B = R_P$ is a ϕ-CR for some nonmaximal prime ideal P of R.*
2. *$IB = B$ for some proper ideal I of R.*
3. *$1/s \in B$ for some nonzerodivisor $s \in M$.*

Proof. (1)⇒ (2). No comments.
(2)⇔ (3). This is clear by [10, Proposition 3.6].
(3)⇒ (1). Suppose that B contains an element of the form $1/s$ for some nonzerodivisor $s \in M$. Then by [10, Proposition 3.8] B is a ϕ-CR, and hence is quasilocal. Thus, let N be the maximal ideal of B, and let $P = N \cap R$. Since $s \notin P$, P is a nonmaximal prime ideal of R. Clearly, $Z(R) \subset P$. Hence, $R_P \subset B$. Now, let $x \in B \setminus R$. If $x^{-1} \in R$, then $x = 1/d$ for some $d \in R \setminus P$. Thus, $x \in R_P$. Thus, assume that $x^{-1} \notin R$. Hence, $xs = m \in M$ by Lemma 3.1. Thus, $x = m/s \in R_P$. Hence, $B \subset R_P$. □

The following result is a generalization of [10, Theorem 3].

COROLLARY 3.5. *Let R be a ϕ-PVR with maximal ideal M, and let B be an overring of R such that B is a ϕ-CR with maximal ideal N. If $P = N \cap R \neq M$, then $B = R_P$.*

Proof. Since $P \neq M$, B contains an element of the form $1/s$ for some nonzerodivisor $s \in M$. Hence, by the above proposition, the proof is complete. □

The proof of the following result is similar to that in [4, Theorem 2.1]. Hence, we invite the reader to finish the proof.

PROPOSITION 3.6. *Let R be a ϕ-PVR with maximal ideal M and $u \in (M : M) \setminus R$. Then $R[u]$ is a ϕ-PVR if and only if $R[u]$ is quasilocal. Furthermore, if $R[u]$ is quasilocal for some $u \in (M : M) \setminus R$, then $R[u]$ is a ϕ-PVR with maximal ideal M.*

PROPOSITION 3.7. *Let R be a ϕ-PVR with maximal ideal M. If C is an overring of R such that C does not contain an element of the form $1/s$ for some nonzerodivisor $s \in M$, then $C \subset (M : M)$.*

Proof. Let $x \in C \setminus R$. By hypothesis, $x^{-1} \notin R$. Hence, $xM \subset M$ by Lemma 3.1. Thus, $x \in (M : M)$. □

COROLLARY 3.8. *Let R be a ϕ-PVR with maximal ideal M. Then every overring of R is a ϕ-PVR if and only if $R[u]$ is quasilocal for each $u \in (M : M) \setminus R$.*

Proof. Suppose that $R[u]$ is quasilocal for each $u \in (M : M)\setminus R$. Let C be an overring of R. If C contains an element of the form $1/s$ for some nonzerodivisor $s \in M$, then C is a ϕ-PVR by Proposition 3.4. Hence, assume that C does not contain an element of the form $1/s$ for some nonzerodivisor $s \in M$. Hence, $C \subset (M : M)$ by Proposition 3.7. Let $u \in C \setminus R$. Then $R[u]$ is quasilocal by hypothesis. Hence, by Proposition 3.6, M is the maximal ideal of $R[u]$. Thus, $u^{-1} \in R[u] \subset C$. Hence, M is the maximal ideal of C. Thus, by applying Proposition 1.1(5) to the ring C, we conclude that C is a ϕ-PVR. □

We recall the following result.

LEMMA 3.9. [10, Proposition 2.3] *A ring R is a ϕ-CR if and only if $Nil(R)$ is a divided prime ideal of R and for every $a, b \in R \setminus Nil(R)$, either $a \mid b$ in R or $b \mid a$ in R.*

Recall that an ideal of R is called regular if it contains a nonzerodivisor of R. If every regular ideal of R is generated by its set of nonzerodivisors, then R is called a *Marot ring.* Also, recall that a ring R *has few zerodivisors* if $Z(R)$ is a finite union of prime ideals. We have the following result which is a generalization of [10, Proposition 6].

PROPOSITION 3.10. *Let R be a ϕ-PVR. Then :*

1. *R is a Marot ring.*
2. *If $R \neq T(R)$, then $T(R)$ is a ϕ-CR.*

Proof. **(1).** Since $Z(R)$ is a prime ideal of R by Proposition 1.1(6), R has few zerodivisors. Hence, R is a Marot ring by [17, Theorem 7.2].
(2). Since $Nil(R)$ is a divided prime ideal of R, $Nil(T(R)) = Nil(R)$. Now, let $x, y \in T(R) \setminus Nil(R)$. Then $x = a/s$ and $y = b/s$ for some $a, b \in R \setminus Nil(R)$ and $s \in R \setminus Z(R)$. By Lemma 3.9, we need to show that either $x \mid y$ in $T(R)$ or $y \mid x$ in $T(R)$. If $a \mid b$ in R, then $x \mid y$ in $T(R)$. Hence, assume that $a \nmid b$ in R. Since R is a ϕ-PVR and $R \neq T(R)$, $b \mid ad$ in R for some $d \in M \setminus Z(R)$. Thus, $ad = bc$ for some $c \in R$. Thus, $a/s = (b/s)(c/d)$. Thus, $y \mid x$ in $T(R)$. □

REMARK 3.11. *Let R be a ϕ-PVR with maximal ideal M such that M contains a nonzerodivisor of R, and let I be a proper ideal of R. Then, since $V = (M : M)$ is a ϕ-CR with maximal ideal M, it is easy to see that there is a ϕ-CR V between R and $T(R)$ such that $IV \neq V$.*

The proof of the following result starts exactly as in [18, Theorem 56].

THEOREM 3.12. *Let R be a ϕ-PVR with maximal ideal M such that M contains a nonzerodivisor of R, let C be an overring of R ($R \subset C \subset T(R)$), and let I be a proper ideal of C. Then there exists a ϕ-CR B such that $C \subset B \subset T(R)$ and $IB \neq B$.*

Proof. Consider all pairs (C_α, I_α), where C_α is a ring between C and $T(R)$, and $I_\alpha \neq C_\alpha$, $I \subset I_\alpha$. We partially order the pairs by decreeing inclusion to mean both $C_\alpha \supset C_\beta$ and $I_\alpha \supset I_\beta$. Zorn's Lemma is applicable to yield a maximal pair (B, J). To show that B is a ϕ-CR, by Lemma 3.9, we only need to show that $Nil(B)$ is a divided prime ideal of B and for every $a, b \in B$ either $a \mid b$ in B or $b \mid a$ in B. Clearly, $IB \neq B$, $C \subset B \subset T(T)$, and $Nil(B) = Nil(R)$ is a divided prime ideal of B. Let $x \in T(R) \setminus R$. Since R is a divided ring by Proposition 1.1(2) and $x \notin R$, $x = a/b$ for some nonzerodivisors a, b of R. Hence, x is a unit in $T(R)$. Thus, $JB[x] \neq B[x]$ or $JB[x^{-1}] \neq B[x^{-1}]$ by [18, Theorem 55]. Hence, by the maximality of the pair (B, J), either $x \in B$ or $x^{-1} \in B$. Thus, if $x, y \in B \setminus R$, then $x \mid y$ in B or $y \mid x$ in B. Now, let $a, b \in R$ and suppose that $a \nmid b$ in R. Since R is a ϕ-PVR and M contains a nonzerodivisor of R, $b \mid as$ for some nonzerodivisor $s \in M$. Thus, $as = bc$ for some $c \in M$. Suppose that $c \in Z(R)$. Since $Z(R)$ is a divided prime ideal of R and $s \notin Z(R)$, $s \mid c$ in R. Hence, $b \mid a$ in R and therefore $b \mid a$ in B. Now, assume that $c \notin Z(R)$. If $s \mid c$ in R, then, once again, $b \mid a$ in R and we are done. Thus, suppose that $s \nmid c$ in R. Then $x = c/s \in T(R) \setminus R$, and hence either $x \in B$ or $x^{-1} \in B$ as we have shown earlier in the proof. Thus, either $b \mid a$ in B or $a \mid b$ in B. Finally, suppose that $a \in R$ and $b \in B \setminus R$. Write $b = c/d$ for some $c \in R$ and $d \in R \setminus Z(R)$. Since $Z(R)$ is a divided ideal of R by Proposition 1.1(6) and $b = c/d \notin R$, we conclude that $c \in R \setminus Z(R)$. If $a \in Z(R)$, then $c \mid a$ in R and hence $b \mid a$ in B. Thus, assume that $a \notin Z(R)$. Let $x = ad/c$. If $x \in R$, then $b \mid a$ in B. Otherwise, $x \in T(R) \setminus R$. Hence, either $x \in B$ or $x^{-1} \in B$ as we have shown earlier in the proof. Thus, either $b \mid a$ in B or $a \mid b$ in B. Hence, B is a ϕ-CR by Lemma 3.9. □

PROPOSITION 3.13. *Let R be a ϕ-PVR and B be an overring of R such that B is a ϕ-CR. Then $R' \subset B$.*

Proof. Deny. Then there is an $x \in R' \setminus B$. Hence, since R' is a ϕ-PVR with maximal ideal M by Proposition 3.3, x is a unit in R'. Since $x \notin B$ and B is a ϕ-CR, $x^{-1} \in B$. Since $x \in R'$, $x \in R[x^{-1}]$ by [18, Theorem 15]. Hence, $x \in R[x^{-1}] \subset B$, which is a contradiction. Thus, $R' \subset B$. □

THEOREM 3.14. *Let R be a ϕ-PVR with maximal ideal M such that M contains a nonzerodivisor. Then R' is the intersection of all the ϕ-CRs between R and $T(R)$.*

Proof. By Proposition 3.13, R' is contained in the intersection of all the ϕ-CRs between R and $T(R)$. Let $y \in$ the intersection of all the ϕ-CRs between R and $T(R)$; we must show that $y \in R'$. Suppose not. By [18, Theorem 15], $y \notin C = R[y^{-1}]$. Let $I = y^{-1}C$. Then I is a proper ideal of C. By Theorem 3.12 there is a ϕ-CR B between C and $T(R)$ such that $IB \neq B$. But by hypothesis $y \in B$, and we have our contradiction. □

The following result is a generalization of [12, Theorem 8].

THEOREM 3.15. *Let R be a ϕ-PVR with maximal ideal M. Then every overring of R is a ϕ-PVR if and only if every ϕ-CR between R and $T(R)$ other than $(M : M)$ is of the form R_P for some nonmaximal prime ideal P of R.*

Proof. If $T(R) = R$, then there is nothing to prove. Hence, assume that M contains a nonzerodivisor of R. Suppose that every overring of R is a ϕ-PVR. Then $R' = (M : M)$ by [8, Proposition 15(1)]. Let C be an overring of R such that $C \neq (M : M)$ and C is a ϕ-CR. Since every overring of R not containing an element of the form $1/s$ for some nonzerodivisor s of R is contained in $R' = (M : M)$ by Proposition 3.7 and hence is a ϕ-PVR with maximal ideal M by Proposition 3.3 and $(M : M)$ is the only ϕ-CR between R and $T(R)$ that has maximal ideal M by [10, Lemma 3.1(1)], $C \not\subset R' = (M : M)$. Thus, C must contain an element of the form $1/s$ for some nonzerodivisor $s \in M$. Hence, $C = R_P$ for some nonmaximal prime ideal P of R by Proposition 3.4.

Conversely, suppose that every ϕ-CR between R and $T(R)$ other than $(M : M)$ is of the form R_P for some nonmaximal prime ideal P of R. Then $(M : M)$ is contained in every ϕ-CR between R and $T(R)$. Hence, $(M : M)$ is the intersection of all the ϕ-CRs between R and $T(R)$. Thus, by Theorem 3.14, $R' = (M : M)$. Hence, every overring of R is a ϕ-PVR by [8, Proposition 15(1)]. □

In light of [8, Proposition 15(1)] and the above Theorem, we have the following result.

COROLLARY 3.16. *Let R be a ϕ-PVR with maximal ideal M such that $R' \neq (M : M)$. Then there is a ϕ-CR that is properly contained between R' and $(M : M)$.*

Combining [8, Proposition 15(1)], Proposition 3.3, Proposition 3.4, Proposition 3.8, and Theorem 3.14, we arrive at the following result that is a generalization of ([4, Corollary 2.2], [12, Theorem 8], and [11, Corollary 17]).

COROLLARY 3.17. *Let R be a ϕ-PVR with maximal ideal M. Then the following statements are equivalent:*

1. *Every overring of R is a ϕ-PVR.*
2. *$R[u]$ is a ϕ-PVR for each $u \in (M:M) \setminus R$.*
3. *$R[u]$ is quasilocal for each $u \in (M:M) \setminus R$.*
4. *If B is an overring of R and $B \subset (M:M)$, then B is a ϕ-PVR with maximal ideal M.*
5. *If B is an overring of R and $B \subset (M:M)$, then B is quasilocal.*
6. *Every overring of R is quasilocal.*
7. *Every ϕ-CR between R and $T(R)$ other than $(M:M)$ is of the form R_P for some nonmaximal prime ideal P of R.*
8. *$R' = (M:M)$.*

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